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1983 J. Phys. A: Math. Gen. 16 L413

(http://iopscience.iop.org/0305-4470/16/12/004)

J. Phys. A: Math. Gen. **16** (1983) L413–L417. Printed in Great Britain

LETTER TO THE EDITOR

Time continuous limit for the Baxter model

Valter Líbero and J R Drugowich de Felício

Departamento de Física e Ciência dos Materiais, Instituto de Física e Química de São Carlos—USP, Caixa Postal 369, 13560 São Carlos, SP, Brasil

Received 21 June 1983

Abstract. We obtain the associated Hamiltonian for the symmetric eight-vertex model by taking the time-continuous limit in an equivalent Ashkin–Teller model. The result is a Heisenberg Hamiltonian with coefficients J_x, J_y and J_z identical to those found by Sutherland for choices of the parameters a, b, c and d that bring the model close to the transition.

The change in the operators is accomplished explicitly, the relation between the crossover operator for the Ashkin–Teller model and the energy operator for the eight-vertex model being obtained in a transparent form.

The work of McCoy and Wu (1968), showing that the *XXZ* Hamiltonian commutes with the transfer matrix of the six-vertex model, inspired further research into the possible existence of other similar relations. Two years later, Sutherland (1970) found that for the symmetric eight-vertex model there is also a Hamiltonian commuting with the transfer matrix T_{8V}, namely the Hermitian *XYZ* Hamiltonian

$$H_{XYZ} = -\sum_{j=1}^{N} (J_x S_j^x S_{j+1}^x + J_y S_j^y S_{j+1}^y + J_z S_j^z S_{j+1}^z), \tag{1}$$

with coefficients given by

$$J_x = (ab+cd), \qquad J_y = (ab-cd), \qquad J_z = (a^2+b^2-c^2-d^2)/2, \tag{2a, b, c}$$

where a, b, c and d are the vertex weights (see figure 1). It is worthwhile to mention that the Heisenberg Hamiltonian (1) is exactly the logarithmic derivative of the transfer matrix $T(v, \eta, k)$ of the symmetric eight-vertex (8V) model with respect to v at the value $v = \eta$, i.e.

$$H_{XYZ} = (\partial/\partial v) \log T(v, \eta, k)|_{v=\eta}. \tag{3}$$

Equation (3) was first obtained by Baxter (1972), who used the convenient elliptic

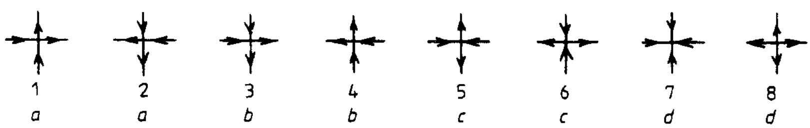

Figure 1. The eight arrow configurations allowed at a vertex with the corresponding vertex weights.

parametrisation for the vertex weights,

$$a:b:c:d = \mathrm{sn}(v+\eta,k):\mathrm{sn}(v-\eta,k):\mathrm{sn}(2\eta,k):k\,\mathrm{sn}(v+\eta,k)\mathrm{sn}(v-\eta,k)\mathrm{sn}(2\eta,k), \tag{4}$$

to solve the model in the principal domain

$$C > a+b+d, \tag{5}$$

with non-negative weights. In particular, he showed that the symmetric 8V model undergoes a phase transition at the border

$$c = a+b+d. \tag{6}$$

In this letter we show that, in the critical region associated with this phase transition, the Hamiltonian in (1) can be derived through an alternative procedure, namely the time-continuous limit (Fradkin and Susskind 1978). This limit can *not* be taken in the Ising representation for the Baxter model (Kadanoff and Wegner 1971, Wu 1971), shown in figure 2, because in this formulation of the model next-nearest-neighbour and strictly isotropic† interactions are involved. To circumvent this difficulty we write the symmetric 8V model in the Ashkin–Teller (1943) representation, which possesses only first-neighbour interactions in well defined directions (see figure 3).

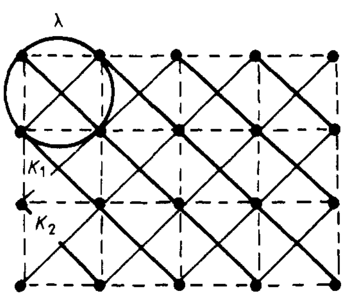

Figure 2. Ising representation for the Baxter model. The four-spin coupling involves first and second neighbours.

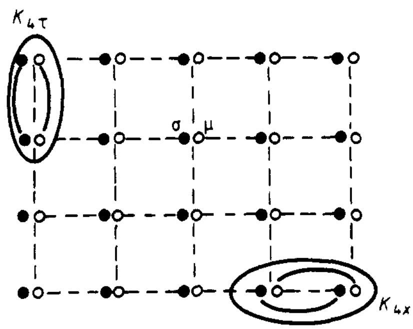

Figure 3. Ising representation for the Ashkin–Teller model. The four-spin coupling involves only first neighbours.

The Ashkin–Teller (AT) model is defined by the action (Fan 1972)

$$-\mathscr{H}/kT = \sum_{r,\alpha} (K_{0\alpha} + K_{1\alpha}\mu(\boldsymbol{r})\mu(\boldsymbol{r}+\hat{e}_\alpha) + K_{2\alpha}\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}+\hat{e}_\alpha)$$
$$+ K_{4\alpha}\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}+\hat{e}_\alpha)\mu(\boldsymbol{r})\mu(\boldsymbol{r}+\hat{e}_\alpha)), \tag{7}$$

where σ and μ are classical Ising variables, $\boldsymbol{r} = (j, k)$ labels the lattice sites and $\hat{e}_\alpha$ are unit vectors in the x (spatial) and τ (temporal) directions. The coupling constants are chosen so that the AT model be equivalent to a normal (non-staggered) 8V model in the medial lattice (Wegner 1972, Wu 1977), as shown in figure 4; the spatial

† The four-spin coupling does not allow us to separate the interactions into temporal and spatial ones.

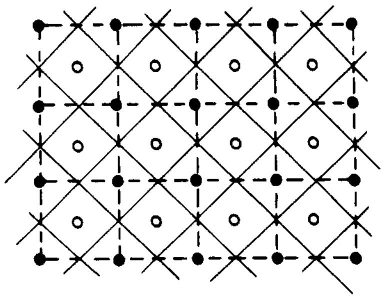

Figure 4. The original lattice for the Ashkin–Teller model (full circles and broken lines) and the medial lattice where lies the equivalent eight-vertex model.

couplings are then given by

$$\exp(4K_{0x}) = (a^2-d^2)(c^2-b^2)/16, \qquad \exp(4K_{1x}) = (a+d)(b+c)/(a-d)(c-b), \tag{8a, b}$$

$$\exp(4K_{2x}) = (a+d)(a-d)/(c+b)(c-b), \qquad \exp(4K_{4x}) = (a+d)(c-b)/(c+b)(a-d), \tag{8c, d}$$

and the temporal (τ) ones by

$$\exp(4K_{0\tau}) = (a^2-c^2)(d^2-b^2)/16, \qquad \exp(4K_{1\tau}) = (a+c)(d+b)/(a-c)(d-b), \tag{9a, b}$$

$$\exp(4K_{2\tau}) = (a+c)(a-c)/(d+b)(d-b), \tag{9c}$$

$$\exp(4K_{4\tau}) = (a+c)(d-b)/(a-c)(d+b). \tag{9d}$$

The transfer matrix for this model is given by

$$T = \exp\Big(\sum_j (K_{0x} + K_{1x}\mu_j^z\mu_{j+1}^z + K_{2x}\sigma_j^z\sigma_{j+1}^z + K_{4x}\sigma_j^z\sigma_{j+1}^z\mu_j^z\mu_{j+1}^z)\Big)$$
$$\times \prod_j \{\exp(K_{0\tau}+K_{1\tau}+K_{2\tau}+K_{4\tau})[\mathbb{1} + \exp(-2K_{1\tau}-2K_{4\tau})\mu_j^x$$
$$+ \exp(-2K_{2\tau}-2K_{4\tau})\sigma_j^x + \exp(-2K_{1\tau}-2K_{2\tau})\sigma_j^x\mu_j^x]\}, \tag{10}$$

where σ_j^x, σ_j^z and μ_j^x, μ_j^z are two sets of Pauli matrices. As a first step towards taking the time-continuous limit we observe that the off-diagonal term of the transfer matrix can be rewritten in an exponential form:

$$\prod_j \{\ldots\} = \exp\Big(\sum_j (K_0^* + K_1^*\mu_j^x + K_2^*\sigma_j^x + K_4^*\sigma_j^x\mu_j^x)\Big), \tag{11}$$

provided that the starred couplings satisfy

$$K_0^* = K_{0x} + \ln 2, \qquad K_1^* = K_{2x}, \tag{12a, b}$$

$$K_2^* = K_{1x}, \qquad K_4^* = K_{4x}. \tag{12c, d}$$

Next, following Fradkin and Susskind (1978) we consider the weak coupling limit

$$K_{ix} \to 0, \qquad i = 1, 2 \text{ or } 4, \tag{13}$$

for which the evolution in the τ direction is described by the Hamiltonian

$$H = -\tau^{-1}\log T = -K_{1x}^{-1}\log T$$
$$= -\sum_j (\mu^z_j\mu^z_{j+1} + \Gamma\sigma^z_j\sigma^z_{j+1} - \Delta\sigma^z_j\sigma^z_{j+1}\mu^z_j\mu^z_{j+1} + \sigma^x_j + \Gamma\mu^x_j - \Delta\sigma^x_j\mu^x_j), \qquad (14)$$

where

$$\Gamma = K_{2x}/K_{1x}, \qquad \Delta = -K_{4x}/K_{1x}. \qquad (15a, b)$$

Equation (14) defines the time-continuous Hamiltonian; however, H is given in an unusual form. To compare with the exact result, equation (1), two further steps are necessary.

(A) Firstly we introduce a duality transformation by defining another set of operators†

$$\eta^x_{j+1/2} = \mu^z_j\mu^z_{j+1}, \qquad \eta^z_{j+1/2} = \prod_{k\leq j}\mu^x_k, \qquad (16a, b)$$

in terms of which the Hamiltonian H (equation (14)) is written

$$H = -\sum_j (\eta^x_{j+1/2} + \Gamma\sigma^z_j\sigma^z_{j+1} - \Delta\sigma^z_j\sigma^z_{j+1}\eta^x_{j+1/2} + \sigma^x_j + \Gamma\eta^z_{j-1/2}\eta^z_{j+1/2} - \Delta\sigma^x_j\eta^z_{j-1/2}\eta^z_{j+1/2}). \qquad (17)$$

(B) Next, having a string of spins, separated by half of the original lattice parameter, which will be henceforth named σ_k ($k = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, N+\frac{1}{2}$), we once again apply the dual transformation to all spins (Kohmoto *et al* 1981):

$$S^x_{j+1/4} = \sigma^z_j\sigma^z_{j+1/2}, \qquad S^z_{j+1/4} = \prod_{k\leq j}\sigma^x_k. \qquad (18a, b)$$

In terms of these operators the Hamiltonian H, after the rotation $S^z \to S^x$ and $S^x \to -S^z$, is

$$H = -\sum_{i=0}^{2N-1} (S^x_iS^x_{i+1} + \Delta S^y_iS^y_{i+1} + \Gamma S^z_iS^z_{i+1}), \qquad (19)$$

which is equivalent to equation (1). As shown in figure 5 we have shrunk to one half the lattice parameter and displaced the origin by one fourth.

The coefficients Γ and Δ on the right-hand side of (19) remain to be analysed. From (8) we find, in the limit $K_{ix} \to 0$,

$$\Gamma = K_{2x}/K_{1x} = (a-c)/(d+b), \qquad \Delta = -K_{4x}/K_{1x} = (b-d)/(d+b), \qquad (20a, b)$$

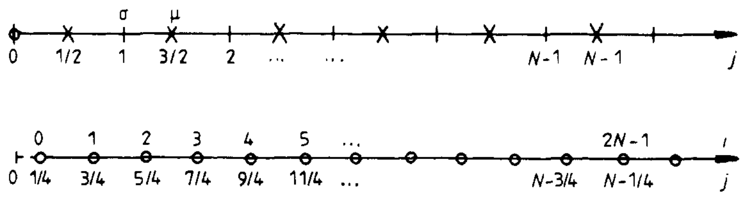

Figure 5. (a) After the first duality transfomation (equation (16)) we have a chain with $2N$ points. (b) Below the axis we have the localisation of the new variables S (equation (18)). To make the notation easier we will use the variable i.

† We are using periodic boundary conditions, so that $\mu^z_{N+1} = \mu^z_1$.

so that, in the limit $a \cong c$ and $b,d \cong 0$†, our results for Γ and Δ are identical to those found by Sutherland (1970). Indeed, from (1) and (2) we have in this approximation

$$\frac{J_z}{J_x}=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}=\frac{(a-c)(a+c)+(b+d)(b-d)}{2[c(b+d)+(a-c)b]}\cong\frac{(a-c)}{(b+d)}=\Gamma, \tag{21a}$$

$$\frac{J_y}{J_x}=\frac{ab-cd}{ab+cd}=\frac{c(b-d)+(a-c)b}{c(b+d)+(a-c)b}\cong\frac{b-d}{b+d}=\Delta. \tag{21b}$$

Finally we remark on the possibility of obtaining, with this procedure, the relation between the energy operator for the 8V model and the crossover operator for the AT model. We start by rewriting the diagonal part on the right-hand side of (10) as

$$\tfrac{1}{2}(K_{1x}+K_{2x})(\mu_j^z\mu_{j+1}^z+\sigma_j^z\sigma_{j+1}^z)+\tfrac{1}{2}(K_{1x}-K_{2x})(\mu_j^z\mu_{j+1}^z-\sigma_j^z\sigma_{j+1}^z)+K_{4x}\sigma_j^z\sigma_{j+1}^z\mu_j^z\mu_{j+1}^z, \tag{22}$$

hence exhibiting explicitly the energy $(\sigma_j^z\sigma_{j+1}^z+\mu_j^z\mu_{j+1}^z)$ and the crossover $(\mu_j^z\mu_{j+1}^z-\sigma_j^z\sigma_{j+1}^z)$ operators for the AT model (Kadanoff and Brown 1979). We can then follow the effect of the transformations (A) and (B) in the crossover operator of the AT model,

$$\mu_j^z\mu_{j+1}^z-\sigma_j^z\sigma_{j+1}^z \xrightarrow{(A)} \eta_{j+1/2}^x-\sigma_j^z\sigma_{j+1}^z \xrightarrow{(B)} S_{2i}^zS_{2i+1}^z-S_{2i}^xS_{2i+1}^x. \tag{23}$$

The AT crossover operator is hence transformed into the 8V energy operator. This result implies that for $K_{4x}<K_{1x}$, in the absence of the crossover operator $(K_{1x}=K_{2x})$, we are at the critical temperature of the Baxter model.

In conclusion we have shown that, using the AT representation of the 8V model, it is possible to obtain the time-continuous Hamiltonian for the Baxter model which coincides, in the critical region, with the result obtained by Sutherland (1970).

We wish to thank L N Oliveira for stimulating discussions and careful reading of the manuscript.

This work was partially supported by the FINEP, CAPES and FAPESP. In addition, one of us (JRDF) acknowledges financial support by the CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico, Brasil).

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† These conditions are obtained by imposing $K_{ix}\to 0$.